

# KIRILLOV–RESHETIKHIN CRYSTALS $B^{1,s}$ USING NAKAJIMA MONOMIALS FOR $\widehat{\mathfrak{sl}}_n$

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**ABSTRACT.** We give a realization of the Kirillov–Reshetikhin crystal  $B^{1,s}$  using Nakajima monomials for  $\widehat{\mathfrak{sl}}_n$  using the crystal structure given by Kashiwara. We describe the tensor product  $\bigotimes_{i=1}^N B^{1,s_i}$  in terms of a shift of indices, allowing us to recover the Kyoto path model. We give a description of the limit of the coherent family of crystals  $\{B^{1,s}\}_{s=1}^\infty$  using Nakajima monomials, which allows us to recover the path model for  $B(\infty)$ . Additionally, we give a model for the KR crystals  $B^{r,1}$  using Nakajima monomials.

## 1. INTRODUCTION

A special class of finite-dimensional modules of the derived subalgebra of the Drinfel’d–Jimbo quantum group  $U'_q(\widehat{\mathfrak{sl}}_n)$  called Kirillov–Reshetikhin (KR) modules have received significant attention over the past 20 years. KR modules have many remarkable properties and deep connections with mathematical physics. For example, KR modules arise in the study of certain solvable lattice models [BBB16, JM95, KP84]. Their characters (resp.  $q$ -characters [FM01, FR99]) satisfy the Q-system (resp. T-system) relations, which come from a certain cluster algebra [DFK09, Her10, Nak03a]. This gives a fermionic formula interpretation and a relation to the string hypothesis in the Bethe ansatz for solving Heisenberg spin chains. The graded characters of (resp. Demazure submodules of) tensor products of certain KR modules, the fundamental representations, are also (resp. nonsymmetric) Macdonald polynomials at  $t = 0$  [LNS<sup>+</sup>14, LNS<sup>+</sup>16b] (resp. [LNS<sup>+</sup>15]).

In the seminal papers [Kas90, Kas91], Kashiwara defined the crystal basis of a representation of a quantum group, which is a basis that is well-behaved in the  $q \rightarrow 0$  limit and affords a combinatorial description. Furthermore, he showed every irreducible highest weight representation admits a crystal basis  $B(\lambda)$ . While KR modules are cyclic modules, they are not highest weight modules. Yet, KR modules for  $U'_q(\widehat{\mathfrak{sl}}_n)$  admit crystal bases [KKM<sup>+</sup>92b] (conjecturally for all affine types [HKO<sup>+</sup>99, HKO<sup>+</sup>02], which is known for non-exceptional types [OS12] and some other special cases [JS10, KMOY07, Yam98]), which are known as Kirillov–Reshetikhin (KR) crystals, and contain even further connections to mathematical physics. For example, KR crystals are in bijection with combinatorial objects that arise naturally from the Bethe ansatz called rigged configurations [KKR86, KR86, KSS02, DS06]. KR crystals  $B^{1,s}$  can be used to model the Takahashi–Satsuma box-ball system [TS90], where rigged configurations are invariants called action-angle variables [KOS<sup>+</sup>06, Tak05]. They are also perfect crystals [FOS10], and therefore, they can be used to construct the Kyoto path model [KKM<sup>+</sup>92a, KKM<sup>+</sup>92b, OSS03], which came from the study of 2D solvable lattice models and Baxter’s corner transfer matrix [Bax89].

Despite intense study, relatively little is understood about KR crystals. In particular, there is currently not a combinatorial model for KR crystals where all crystal operators are given by the same rules, the model is valid for general  $B^{r,s}$ , and the model is given uniformly across all affine types. By using the decomposition into  $U_q(\mathfrak{sl}_n)$ -crystals and the Dynkin diagram automorphism, we can lift the tableaux model of [KN94] to a model for KR crystals for  $U'_q(\widehat{\mathfrak{sl}}_n)$  [Shi02]. However, this process

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obscures the affine crystal operators as it uses the promotion operator of Schützenberger [Sch72], and so it is desirable to have a model where all of the crystal operators are given by the same rules. A similar procedure was utilized in [FOS09, JS10], but using type-dependent information and it fails for type  $E_8^{(1)}$  due to the Dynkin diagram not admitting any non-trivial automorphisms.

Partial progress has been made on this problem. Naito and Sagaki constructed a model uniform across all types for tensor products of  $B^{r,1}$  by using the usual crystal structure on Lakshmibai–Seshadri (LS) paths for level-zero representations and projecting onto the classical weight space [NS03, NS05, NS06b, NS06a, NS08]. There is another description of these paths called quantum LS paths [LNS<sup>+</sup>14, LNS<sup>+</sup>16a]. Lenart and Lubovsky performed a similar construction using a discrete version of quantum LS paths called the quantum alcove path model [LL15]. Yet, it is not known how to extend these models for general  $B^{r,s}$ . On the other side, models for  $B^{r,s}$  were constructed in [Kus13, Kus16, Kwo13] for type  $A_n^{(1)}$ , but these are not known to extend (uniformly) to other affine types.

There is a  $t$ -analog of  $q$ -characters (or  $q, t$ -characters for short) that was studied by Nakajima [Nak01, Nak03a, Nak03b, Nak04, Nak10]. From this study, Nakajima gave a  $U_q(\mathfrak{sl}_n)$ -crystal structure on the monomials that appear in the  $q$ -character [Nak03b]. Based on this model, Kashiwara [Kas03] independently constructed a different crystal structure on the  $q$ -character monomials. These two crystal structures were later simultaneously generalized by Sam and Tingley [ST14], where a connection to quiver varieties was also made. This is known as the Nakajima monomial model. We note that Kashiwara’s crystal structure works for  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystals  $B(\lambda)$ , but Nakajima’s is only valid when  $n$  is odd.

For an extremal level-zero crystal  $B(\lambda)$  (so  $\lambda$  is a level-zero weight), there exists an automorphism  $\kappa$  such that  $B(\lambda)/\kappa \cong \bigotimes_{i=1}^N B^{r_i,1}$  as  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystals, where  $\lambda = \sum_{i=1}^N \Lambda_{r_i}$ . This was the construction of Naito and Sagaki previously mentioned, where the description was given explicitly in terms of LS paths. A similar construction was given for Nakajima monomials by Hernandez and Nakajima [HN06].

Nakajima’s  $q, t$ -characters have also been well-studied using a variety of techniques. While their definition is combinatorial, Nakajima used quiver varieties to show their existence in simply-laced types [Nak04]. Hernandez reformulated the definition to be purely algebraic by using a  $t$ -analog of screening operators [Her04]. Nakajima also showed that  $q, t$ -characters can be used to determine the change of basis from standard to simple  $U_q(\widehat{\mathfrak{sl}}_n)$ -modules in the Grothendieck group [Nak01]. From this, Kodera and Naoi showed that the graded decomposition of a standard modules in terms of simple modules categorifies the graded decomposition into  $U_q(\mathfrak{sl}_n)$ -modules of a tensor product of fundamental representations [KN12]. The graded decomposition polynomials in this case are also Kostka polynomials [KSS02] and are a specialization of Macdonald polynomials at  $t = 0$ . The  $q, t$ -characters are also related to the natural Jordan filtration of  $\ell$ -weight spaces by the Heisenberg subalgebra of  $U_q(\widehat{\mathfrak{sl}}_n)$  [Zeg15]. Furthermore, Qin used a slight variation of  $q, t$ -characters to realize the generic basis, the dual PBW basis, and the canonical basis of a quantum cluster algebra [Qin14].

Cluster algebras [FZ02] also have strong connections to characters of KR crystals and Nakajima monomials. Hernandez and Leclerc gave an algorithm to compute  $q$ -characters as cluster variables of a cluster algebra from a certain semi-infinite quiver [HL16]. For a double Bruhat cell  $G^{u,v} = BuB \cap B_-vB_-$ , the coordinate ring  $\mathbb{C}[G^{u,v}]$  is an upper cluster algebra whose generalized minors are the cluster variables [BFZ05]. Kanakubo and Nakashima showed that the generalized minors of  $G^{u,e}$  can be expressed as the sum over the Nakajima monomials in a Demazure subcrystal [KN15]. A further connection between cluster variables in  $\mathbb{C}[G^{u,u^{-1}}]$ , with  $u$  a Coxeter element, and representation theory was given by Rupel, Stella, and Williams in [RSW16]. In particular, they show the regular cluster variables in the coordinate ring of the loop group of  $SL_n$  are restrictions of generalized minors of level-zero representations. The specialization of nonsymmetric Macdonald polynomials at  $t = \infty$  can also be described as a Demazure submodule of a tensor product of fundamental representations [NNS15], which satisfy the quantum Q-system relations of [DFK15, DFK16].

All of this suggests that there should exist a natural description of tensor products of KR crystals in terms of Nakajima monomials. Indeed,  $q, t$ -characters are given as the sum over Nakajima monomials graded by energy, which admit a classical crystal structure. Therefore, it is evidence that the entire  $q, t$ -character could be constructed by adding 0-arrows to get a tensor product of KR crystals.

The main result of this paper is a model for the KR crystal  $B^{1,s}$  in type  $A_n^{(1)}$  using Nakajima monomials. From this construction, we are able to describe the tensor product of KR crystals  $\bigotimes_{i=1}^N B^{1,s_i}$  using only Nakajima monomials (*i.e.*, no tensor products). Furthermore, we are able to recover the Kyoto path model. From this construction, we are able to relate the models of [ST14, Tim08] with the Kyoto path model. We also extend our construction to give a Nakajima monomial description of the coherent limit  $B_\infty$ , where we recover the analog of the Kyoto path model for  $B(\infty)$  and the characterization of  $B(\infty)$  given in [KKS07, Thm. 5.1].

The results of our paper suggest a crystal interpretation for the fusion construction of [KKM<sup>+</sup>92a, KKM<sup>+</sup>92b]. Indeed, the kernel of the  $R$ -matrix is approximately given by a commutator relation on the elements of  $B^{1,1}$ , which are uniquely determined by the variables  $X_{i,k}$ . By considering the tensor product as multiplication, we can relate our construction with the kernel of the  $R$ -matrix. Furthermore, our construction gives an explanation of the link between the Nakajima monomial model, the abacus model, multipartition model, and quiver varieties that was explored in [ST14, Tim08, Tim10]. While our model does not naturally extend to general  $B^{r,s}$  or to other affine types, there is evidence that our construction can be modified to the general case.

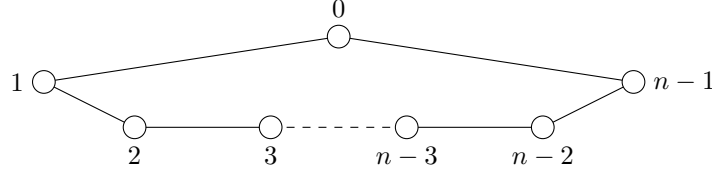
We now give one potential application of our results. To do so, we recall that geometric crystals were introduced by Berenstein and Kazhdan [BK00, BK07], where the  $U_q(\mathfrak{sl}_n)$ -crystal structure of  $B(\infty)$  is lifted to actions on algebraic varieties. This was generalized to a lifting of certain  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystals using Schubert varieties [Nak05]. Nakashima has lifted Nakajima monomials to describe the decoration function of geometric crystals (specifically as generalized minors) in [Nak14] and made a connection to the polyhedral model in [Nak13]. There is also a lifting of the  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystal  $B_\infty$  to the geometric setting given in [KNO08] and the coherent limits of  $\{B^{r,s}\}_{s=1}^\infty$  in [MN16]. The geometric  $R$ -matrix has also been studied [KNO10, Yam01], which was then used to relate a quotient of the liftings of  $B_\infty$  to the unipotent loop group in [LP11].

We expect that our results can be lifted to a statement on geometric crystals, connecting the results of Nakashima with the work on geometric analogs of  $B_\infty$ . Furthermore, we believe our results could be used to construct a geometric lifting of the path model embeddings and give a geometric lifting of highest weight  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystals. We also believe that our results could give a connection to the cluster algebra (geometric)  $R$ -matrices that were recently introduced in [ILP16].

We also consider our results as evidence of a deep connection between cluster algebras,  $q, t$ -characters, and KR crystals. Indeed, KR modules could be considered as loop group representations and are closely related to level-zero representations as mentioned above. Furthermore, after removing certain 0-arrows, KR crystals are Demazure subcrystals of affine highest weight crystals [ST12]. Therefore, we believe that the Nakajima monomials appearing in a realization of general KR crystals to give a connection between the work of [HL16, KN15, Nak05, Nak14, RSW16].

This paper is organized as follows. In Section 2, we give a background on crystals, KR crystals, Nakajima monomials, and the Kyoto path model. In Section 3, we construct a model for  $B^{1,s}$  using Nakajima monomials. In Section 4, we give a method to construct tensor products of KR crystals as a map on Nakajima monomials and relate our construction to the Kyoto path model. In Section 5, we describe the crystal corresponding to the coherent limit of  $\{B^{1,s}\}_{s=1}^\infty$ . In Section 6, we describe the relationship between our model and other models for highest weight  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystals, generalizations of our model to  $B^{r,1}$ , and possible extensions to other types.

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FIGURE 1. Dynkin diagram of  $\widehat{\mathfrak{sl}}_n$ .

## 2. BACKGROUND

In this section, we provide the necessary background.

**2.1. Crystals.** Let  $\widehat{\mathfrak{sl}}_n$  be the affine Kac–Moody Lie algebra of type  $A_{n-1}^{(1)}$  with index set  $I = \{0, 1, \dots, n-1\}$ , Cartan matrix  $(a_{ij})_{i,j \in I}$ , simple roots  $\{\alpha_i\}_{i \in I}$ , simple coroots  $\{h_i\}_{i \in I}$ , fundamental weights  $\{\Lambda_i\}_{i \in I}$ , weight lattice  $P = \text{span}_{\mathbb{Z}}\{\Lambda_i \mid i \in I\}$ , dual weight lattice  $P^\vee$ , canonical pairing  $\langle \cdot, \cdot \rangle: P^\vee \times P \rightarrow \mathbb{Z}$  given by  $\langle h_i, \alpha_j \rangle = a_{ij}$ , and quantum group  $U_q(\widehat{\mathfrak{sl}}_n)$ . See Figure 1 for the Dynkin diagram of  $\widehat{\mathfrak{sl}}_n$ . Let  $P^+ = \text{span}_{\mathbb{Z}_{\geq 0}}\{\Lambda_i \mid i \in I\}$  denote the dominant integral weights. Note that  $\mathfrak{sl}_n$  is the canonical simple Lie algebra given by the index set  $I_0 = I \setminus \{0\}$ . Let  $\overline{\Lambda}_i$  denote the natural projection of  $\Lambda_i$  onto the weight lattice  $\overline{P}$  of  $\mathfrak{sl}_n$ .

Let  $c = h_0 + h_1 + \dots + h_{n-1}$  denote the canonical central element of  $\widehat{\mathfrak{sl}}_n$ . We define the level of a weight  $\lambda$  as  $\langle c, \lambda \rangle$ . Let  $P_s^+ := \{\lambda \in P^+ \mid \langle c, \lambda \rangle = s\}$  denote the set of level  $s$  weights.

We write  $U'_q(\widehat{\mathfrak{sl}}_n) = U_q([\widehat{\mathfrak{sl}}_n, \widehat{\mathfrak{sl}}_n])$ , and let  $\delta = \alpha_0 + \alpha_1 + \dots + \alpha_{n-1}$  denote the null root. Note that the  $U'_q(\widehat{\mathfrak{sl}}_n)$  fundamental weights and simple roots are also given by  $\{\Lambda_i\}_{i \in I}$  and  $\{\alpha_i\}_{i \in I}$ , respectively, but are considered in the weight lattice  $P/\mathbb{Z}\delta$ .

An *abstract  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystal* is a set  $B$  with *crystal operators*  $e_i, f_i: B \rightarrow B \sqcup \{0\}$  for  $i \in I$ , statistics  $\varepsilon_i, \varphi_i: B \rightarrow \mathbb{Z} \sqcup \{-\infty\}$ , and *weight function*  $\text{wt}: B \rightarrow P$  that satisfy the following conditions for all  $i \in I$ :

- (1)  $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle$  for all  $b \in B$  and  $i \in I$ ;
- (2) if  $e_i b \neq 0$  for  $b \in B$ , then
  - (a)  $\varepsilon_i(f_i b) = \varepsilon_i(b) - 1$ ,
  - (b)  $\varphi_i(f_i b) = \varphi_i(b) + 1$ ,
  - (c)  $\text{wt}(f_i b) = \text{wt}(b) + \alpha_i$ ;
- (3) if  $f_i b \neq 0$  for  $b \in B$ , then
  - (a)  $\varepsilon_i(f_i b) = \varepsilon_i(b) + 1$ ,
  - (b)  $\varphi_i(f_i b) = \varphi_i(b) - 1$ ,
  - (c)  $\text{wt}(f_i b) = \text{wt}(b) - \alpha_i$ ;
- (4)  $f_i b = b'$  if and only if  $b = e_i b'$  for  $b, b' \in B$  and  $i \in I$ ;
- (5) if  $\varphi_i(b) = -\infty$  for  $b \in B$ , then  $e_i b = f_i b = 0$ .

Define

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b) \Lambda_i, \quad \varphi(b) = \sum_{i \in I} \varphi_i(b) \Lambda_i,$$

We say an element  $b \in B$  is *highest weight* if  $e_i b = 0$  for all  $i \in I$ . If  $B$  is a  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystal, then we say  $b \in B$  is *classically highest weight* if  $e_i b = 0$  for all  $i \in I_0$ . We say an abstract  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystal is *regular* if

$$\varepsilon_i(b) = \max\{k \mid e_i^k b \neq 0\}, \quad \varphi_i(b) = \max\{k \mid f_i^k b \neq 0\}.$$

**Remark 2.1.** The term regular is sometimes called *seminormal* in the literature.

We call an abstract  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystal  $B$  a  *$U_q(\widehat{\mathfrak{sl}}_n)$ -crystal* if  $B$  is the crystal basis of some  $U_q(\widehat{\mathfrak{sl}}_n)$ -module.

Kashiwara showed in [Kas91] that the irreducible highest weight  $U_q(\mathfrak{g})$ -module  $V(\lambda)$  admits a crystal basis, where  $\mathfrak{g}$  is a symmetrizable Kac–Moody Lie algebra and  $\lambda$  is a dominant integrable weight. We denote this crystal basis by  $B(\lambda)$ , and let  $u_\lambda \in B(\lambda)$  denote the unique highest weight element, which is the unique element of weight  $\lambda$ . There is also an analog of the crystal corresponding to the lower half of  $U_q(\mathfrak{g})$  denoted by  $B(\infty)$ , and let  $u_\infty$  denote the highest weight element of  $B(\infty)$ .

We define the *tensor product* of abstract  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystals  $B_1$  and  $B_2$  as the crystal  $B_2 \otimes B_1$  that is the Cartesian product  $B_2 \times B_1$  with the crystal structure

$$\begin{aligned} e_i(b_2 \otimes b_1) &= \begin{cases} e_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) > \varphi_i(b_1), \\ b_2 \otimes e_i b_1 & \text{if } \varepsilon_i(b_2) \leq \varphi_i(b_1), \end{cases} \\ f_i(b_2 \otimes b_1) &= \begin{cases} f_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \\ b_2 \otimes f_i b_1 & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1), \end{cases} \\ \varepsilon_i(b_2 \otimes b_1) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle), \\ \varphi_i(b_2 \otimes b_1) &= \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle), \\ \text{wt}(b_2 \otimes b_1) &= \text{wt}(b_2) + \text{wt}(b_1). \end{aligned}$$

**Remark 2.2.** Our tensor product convention is opposite of Kashiwara [Kas91].

Let  $B_1$  and  $B_2$  be two abstract  $U_q(\mathfrak{g})$ -crystals. A *crystal morphism*  $\psi: B_1 \rightarrow B_2$  is a map  $B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$  with  $\psi(0) = 0$  such that the following properties hold for all  $b \in B_1$ :

- (1) If  $\psi(b) \in B_2$ , then  $\text{wt}(\psi(b)) = \text{wt}(b)$ ,  $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$ , and  $\varphi_i(\psi(b)) = \varphi_i(b)$ .
- (2) We have  $\psi(e_i b) = e_i \psi(b)$  if  $\psi(e_i b) \neq 0$  and  $e_i \psi(b) \neq 0$ .
- (3) We have  $\psi(f_i b) = f_i \psi(b)$  if  $\psi(f_i b) \neq 0$  and  $f_i \psi(b) \neq 0$ .

An *embedding* and *isomorphism* is a crystal morphism such that the induced map  $B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$  is an embedding and bijection respectively. A crystal morphism is *strict* if it commutes with all crystal operators.

**2.2. Nakajima monomials.** Next, we recall the Nakajima monomial realization of crystals following [ST14].

Let  $\mathcal{M}$  denote the set of Laurent monomials in the commuting variables  $\{Y_{i,k}\}_{i \in I, k \in \mathbb{Z}}$ . Fix an integer  $K$ , and then fix integers  $c_{i,i+1}$  and  $c_{i+1,i}$  such that  $K = c_{i,i+1} + c_{i+1,i}$  for all  $i \in I$ , where all indices are taken mod  $n$ . For a monomial  $m = \prod_{i \in I} \prod_{k \in \mathbb{Z}} Y_{i,k}^{y_{i,k}}$ , define

$$\begin{aligned} \varepsilon_i(m) &= -\min_{k \in \mathbb{Z}} \sum_{s > k} y_{i,s}, & k_e(m) &= \max \left\{ k \mid \varepsilon_i(m) = -\sum_{s > k} y_{i,s} \right\}, \\ \varphi_i(m) &= \max_{k \in \mathbb{Z}} \sum_{s \leq k} y_{i,s}, & k_f(m) &= \min \left\{ k \mid \varphi_i(m) = \sum_{s \leq k} y_{i,s} \right\}, \\ \text{wt}(m) &= \sum_{\substack{i \in I \\ k \in \mathbb{Z}}} y_{i,k} \Lambda_i \end{aligned}$$

Define the crystal operators  $e_i, f_i: \mathcal{M} \rightarrow \mathcal{M} \sqcup \{0\}$  by

$$\begin{aligned} e_i(m) &= \begin{cases} 0 & \text{if } \varepsilon_i(m) = 0, \\ mA_{i,k_e(m)-K} & \text{if } \varepsilon_i(m) > 0, \end{cases} \\ f_i(m) &= \begin{cases} 0 & \text{if } \varphi_i(m) = 0, \\ mA_{i,k_f(m)-K}^{-1} & \text{if } \varphi_i(m) > 0, \end{cases} \end{aligned}$$

where

$$A_{i,k} = Y_{i,k} Y_{i,k+K} Y_{i-1,k+c_{i,i-1}}^{-1} Y_{i+1,k+c_{i,i+1}}^{-1}.$$

We note that the crystal structure of Kashiwara [Kas03] is when  $K = 1$ , and that of Nakajima [Nak03b] is when  $c_{i,i+1} = c_{i+1,i} = 1$  for all  $i \in I$ . Note that in the case of Nakajima, we prohibit odd length cycles in the Dynkin diagram; that is, we can only consider types  $\widehat{\mathfrak{sl}}_{2k+1}$ .

Let  $\mathcal{M}(m)$  denote the closure of  $m$  under the crystal operators  $e_i$  and  $f_i$  for all  $i \in I$ .

**Theorem 2.3** ([ST14]). *Let  $\lambda \in P^+$ , then*

$$\mathcal{M}\left(\prod_{i \in I} Y_{i,k_i}^{\langle h_i, \lambda \rangle}\right) \cong B(\lambda),$$

for any  $(k_i)_{i \in I}$ .

For convenience, we define

$$Y_\lambda := \prod_{i \in I} Y_{i,0}^{\langle h_i, \lambda \rangle},$$

and we denote  $\mathbf{1} = Y_0$ , where  $0 \in P^+$ . We also define  $\mathcal{M}(\lambda) := \mathcal{M}(Y_\lambda)$ .

We modify the definition of  $k_f(m)$  to only be the *finite* partial sums  $0 \leq s \leq k$ :

$$k_f^\dagger(m) = \min \left\{ k \left| \varphi_i(m) = \sum_{0 \leq s \leq k} y_{i,s} \right. \right\}.$$

Next, we define the modified crystal operator

$$f_i^\dagger(m) = mA_{i,k_f^\dagger(m)-K}^{-1},$$

and then define  $\mathcal{M}(\infty)$  as the closure of  $\mathbf{1}$  under  $f_i^\dagger$ .

**Theorem 2.4** ([KKS07]). *Suppose  $c_{ij} \in \mathbb{Z}_{\geq 0}$  and  $K = 1$ . We have*

$$\mathcal{M}(\infty) \cong B(\infty).$$

There is also another set of variables

$$X_{i,k} := Y_{i-1,k+1}^{-1} Y_{i,k} \tag{2.1}$$

introduced in [KKS07], which was used in their description of  $\mathcal{M}(\infty)$  in type  $A_n^{(1)}$ .

Unless otherwise stated, we consider  $c_{ij} = 1$  if  $(i, j) = (n, 0)$  or  $i < j$  when  $(i, j) \neq (0, n)$  and  $c_{ij} = 0$  otherwise (hence  $K = 1$ ). Note that this corresponds to orienting the Dynkin diagram into an ordered cycle, where we draw an arrow  $i \rightarrow i+1$  implying  $c_{i,i+1} = 1$  and the other values  $c_{i,j}$  for  $j \neq i \pm 1$  do not affect the crystal structure.

**2.3. Kirillov–Reshetikhin crystals.** A *Kirillov–Reshetikhin (KR) module*  $W^{r,s}$ , where  $r \in I_0$  and  $s \in \mathbb{Z}_{>0}$ , is a particular irreducible finite-dimensional  $U'_q(\widehat{\mathfrak{sl}}_n)$ -module that has many remarkable properties. KR modules are classified by their Drinfel'd polynomials, and  $W^{r,s}$  is the minimal affinization of the highest weight  $U_q(\mathfrak{sl}_n)$ -representation  $V(s\bar{\Lambda}_r)$  [CP95, CP98]. In particular, it was shown in [KKM<sup>+</sup>92b] that the KR module  $W^{r,s}$  admits a crystal basis  $B^{r,s}$  called a *Kirillov–Reshetikhin (KR) crystal*. Another property is that the KR crystal  $B^{r,s}$  is a *perfect crystal of level  $s$* , which means it satisfies the following conditions:

- (1)  $B^{r,s} \otimes B^{r,s}$  is connected.
- (2)  $\text{wt}(b) \in s\bar{\Lambda}_r + \sum_{i \in I_0} \mathbb{Z}_{\leq 0} \alpha_i$  for all  $b \in B^{r,s}$ .
- (3)  $\langle c, \varepsilon(b) \rangle \geq s$  for all  $b \in B^{r,s}$ .
- (4) For all  $\lambda \in P_s^+$ , there exists unique elements  $b_\lambda, b^\lambda \in B^{r,s}$  such that

$$\varepsilon(b_\lambda) = \lambda = \varphi(b^\lambda).$$

Additionally, we have  $B^{r,s} \cong B(s\overline{\Lambda}_r)$  as  $U_q(\mathfrak{sl}_n)$ -crystals. Next, recall that the Dynkin diagram automorphism  $i \mapsto i+1 \pmod n$  induces a (twisted)  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystal isomorphism  $\text{pr}: B^{r,s} \rightarrow B^{r,s}$  called the *promotion isomorphism* [Shi02]. On semistandard tableaux [KN94], the map  $\text{pr}$  is the (weak) promotion operator of Schützenberger [Sch72]. Hence,  $B^{r,s}$  is a regular crystal, and we define the remaining crystal structure on  $B^{r,s}$  by

$$\begin{aligned} e_0(b) &= \text{pr}^{-1} \circ e_1 \circ \text{pr}, \\ f_0(b) &= \text{pr}^{-1} \circ f_1 \circ \text{pr}, \\ \text{wt}(b) &= \overline{\text{wt}}(b) + k_0 \Lambda_0, \end{aligned}$$

where  $k_0$  is such that  $\langle \text{wt}(b), c \rangle = 0$  (i.e., it is a level 0 weight).

We will be focusing on the KR crystal  $B^{1,s}$ , which is the crystal that naturally corresponds to the *vector representation* of  $W^{1,s}$ . We have

$$B^{1,s} = \left\{ (x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^n x_i = s \right\}$$

with the crystal structure

$$e_i(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } x_{i+1} = 0, \\ (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, x_n) & \text{if } x_{i+1} > 0, \end{cases} \quad (2.2a)$$

$$f_i(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } x_i = 0, \\ (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, x_n) & \text{if } x_i > 0, \end{cases} \quad (2.2b)$$

$$\varepsilon_i(x_1, \dots, x_n) = x_{i+1}, \quad (2.2c)$$

$$\varphi_i(x_1, \dots, x_n) = x_i, \quad (2.2d)$$

$$\text{wt}(x_1, \dots, x_n) = \sum_{i \in I} (x_i - x_{i+1}) \Lambda_i, \quad (2.2e)$$

where all indices are understood mod  $n$ . Note that  $B^{1,s}$  is a regular crystal.

We will also need the *affinization* of a  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystal  $B$ , which is defined as follows. The affinization of  $B$  is the  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystal  $\widehat{B} = \{b(k) \mid b \in B, k \in \mathbb{Z}\}$ , whose crystal structure is given by

$$e_i(b(k)) = \begin{cases} (e_0 b)(k+1) & \text{if } i = 0, \\ (e_i b)(k) & \text{if } i \neq 0, \end{cases}$$

$$f_i(b(k)) = \begin{cases} (f_0 b)(k-1) & \text{if } i = 0, \\ (f_i b)(k) & \text{if } i \neq 0, \end{cases}$$

$$\varepsilon_i(b(k)) = \varepsilon_i(b),$$

$$\varphi_i(b(k)) = \varphi_i(b),$$

$$\text{wt}(b(k)) = \text{wt}(b) + k\delta.$$

We can construct the coherent limit of the family  $\{B^{1,s}\}_{s=1}^\infty$  as follows. The coherent limit is given by

$$B_\infty = \left\{ (x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{Z}, \sum_i x_i = 0 \right\}$$

with the same crystal structure as in Equation (2.2) except

$$e_i(x_1, \dots, x_n) = (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, x_n),$$

$$f_i(x_1, \dots, x_n) = (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, x_n).$$

We note that  $B_\infty$  is generated by  $b_\infty = (0, 0, \dots, 0)$ , but it is not a regular crystal.



**2.4. Kyoto path model.** We recall some of the results of [KKM<sup>+</sup>92a, KKM<sup>+</sup>92b, OSS03], which gives a model for highest weight  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystals using KR crystals.

**Theorem 2.5.** *Let  $\lambda$  be a level  $s$  weight. Let  $B$  be a perfect crystal of level  $s$ . Let  $b^\lambda \in B$  be the unique element such that  $\varphi(b^\lambda) = \lambda$ . Let  $\mu = \varepsilon(b^\lambda)$ . The morphism*

$$\widehat{\Psi}: B(\lambda) \rightarrow \widehat{B} \otimes B(\mu)$$

*defined by  $u_\lambda \mapsto b^\lambda(0) \otimes u_\mu$  is a  $U_q(\widehat{\mathfrak{sl}}_n)$ -crystal isomorphism.*

By iterating  $\widehat{\Psi}$ , we obtain the *Kyoto path model*. For a level  $s$  weight  $\lambda$ , we can construct a model for  $B(\lambda)$  by

$$\widehat{\Psi}^{(+\infty)}: B(\lambda) \rightarrow \widehat{B}^{1,s} \otimes \widehat{B}^{1,s} \otimes \dots \quad (2.3)$$

since  $B^{1,s}$  is a perfect crystal of level  $s$ . Note that to define  $e_i$  for Equation (2.3), we consider  $e_i b = 0$  if it would be otherwise undefined. Furthermore,  $\widehat{\Psi}^{(+\infty)}(u_\lambda)$  is eventually cyclic and, for any  $b \in B(\lambda)$ , the element  $\widehat{\Psi}^{(+\infty)}(b)$  only differs from  $\widehat{\Psi}^{(+\infty)}(u_\lambda)$  in a finite number of factors. Therefore, for any element  $b$  can consider Theorem 2.5 iterated  $N \gg 1$  times (that depends on  $b$ )  $\widehat{\Psi}^{(N)}(b)$  to define the crystal structure on the Kyoto path model using only the KR crystal  $B^{1,s}$ .

We note that there are analogous results for  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystals by considering the branching rule from  $U_q(\widehat{\mathfrak{sl}}_n)$  to  $U'_q(\widehat{\mathfrak{sl}}_n)$ . In particular, there exists a  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystal isomorphism

$$\Psi: B(\lambda) \rightarrow B \otimes B(\mu) \quad (2.4)$$

defined by  $u_\lambda \mapsto b^\lambda \otimes u_\mu$ .

There is an analog of the Kyoto path model for  $B(\infty)$  given by Kang, Kashiwara, and Misra [KKM94] by iterating the following isomorphism.

**Theorem 2.6.** *Let  $B_\infty$  denote the coherent limit of  $\{B_s\}_{s=1}^\infty$ , where  $B_s$  is a perfect crystal of level  $s$ . Let  $b_\infty \in B_\infty$  denote the unique element of weight 0. Then the morphism*

$$\Upsilon: B(\infty) \rightarrow B_\infty \otimes B(\infty)$$

*defined by  $u_\infty \mapsto b_\infty \otimes u_\infty$  is a  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystal isomorphism.*

### 3. MONOMIAL REALIZATION OF $B^{1,s}$

In this section, we describe the construction of  $B^{1,s}$  using Kashiwara's crystal structure of Nakajima monomials.

Define

$$\mathcal{M}^{1,s} := \left\{ \prod_{i=1}^n Y_{i-1,1}^{-x_i} Y_{i,0}^{x_i} \mid x_1, \dots, x_n \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^n x_i = s \right\}.$$

**Theorem 3.1.** *We have*

$$\mathcal{M}^{1,s} \cong B^{1,s}$$

*as  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystals.*

*Proof.* Let  $\Phi: B^{1,s} \rightarrow \mathcal{M}^{1,s}$  be the map

$$(x_1, \dots, x_n) \mapsto \prod_{i=1}^n Y_{i-1,1}^{-x_i} Y_{i,0}^{x_i},$$

and it is clear that  $\Phi$  is a bijection. Thus it remains to show that  $\Phi$  commutes with the crystal operators.

We restrict to the variables only containing  $i$ . It is sufficient to consider

$$m = Y_{i,1}^{k_1} Y_{i,0}^{k_2}$$



with  $k_1 \leq 0 \leq k_2$ . Then

$$\varphi_i(m) = \max \left\{ \sum_{s \leq k} y_{i,s} \mid k \in \mathbb{Z} \right\} = \max\{k_2, k_2 + k_1\} = k_2 \geq 0$$

and

$$\varepsilon_i(m) = \max \left\{ -\sum_{k < s} y_{i,s} \mid k \in \mathbb{Z} \right\} = \max\{-k_2 - k_1, -k_1\} = -k_1 \geq 0$$

since  $k_1 \leq 0 \leq k_2$ .

Hence

$$\begin{aligned} k_f(m) &= \min \left\{ k \mid \varphi_i(m) = \sum_{s \leq k} y_{i,s} \right\} = \min \left\{ k \mid k_2 = \sum_{s \leq k} y_{i,s} \right\} = 0, \\ k_e(m) &= \max \left\{ k \mid \varphi_i(m) = \sum_{s \leq k} y_{i,s} \right\} = \max \left\{ k \mid k_2 = \sum_{s \leq k} y_{i,s} \right\} = 0, \end{aligned}$$

and note that we have used an alternative form of  $k_e$  (see, *e.g.*, [Kas03]). Therefore, we have

$$\begin{aligned} A_{i,k_f}(m) &= A_{i,k_e}(m) = A_{i,0} = Y_{i,0} Y_{i,1} \prod_{j \neq i} Y_{j,c_{ji}}^{(h_j, \alpha_i)} \\ &= Y_{i,0} Y_{i,1} Y_{i-1,c_{i-1,i}}^{-1} Y_{i+1,c_{i+1,i}}^{-1} \\ &= Y_{i,0} Y_{i,1} Y_{i-1,1}^{-1} Y_{i+1,0}^{-1}. \end{aligned}$$

Since  $\varphi_i(m) > 0$  and  $\varepsilon_i(m) > 0$ , we have

$$\begin{aligned} f_i(m) &= A_{i,k_f}^{-1} m = Y_{i,0}^{k_2-1} Y_{i,1}^{k_1-1} Y_{i-1,1} Y_{i+1,0}, \\ e_i(m) &= A_{i,k_e} m = Y_{i,0}^{k_2+1} Y_{i,1}^{k_1+1} Y_{i-1,1}^{-1} Y_{i+1,0}^{-1}. \end{aligned}$$

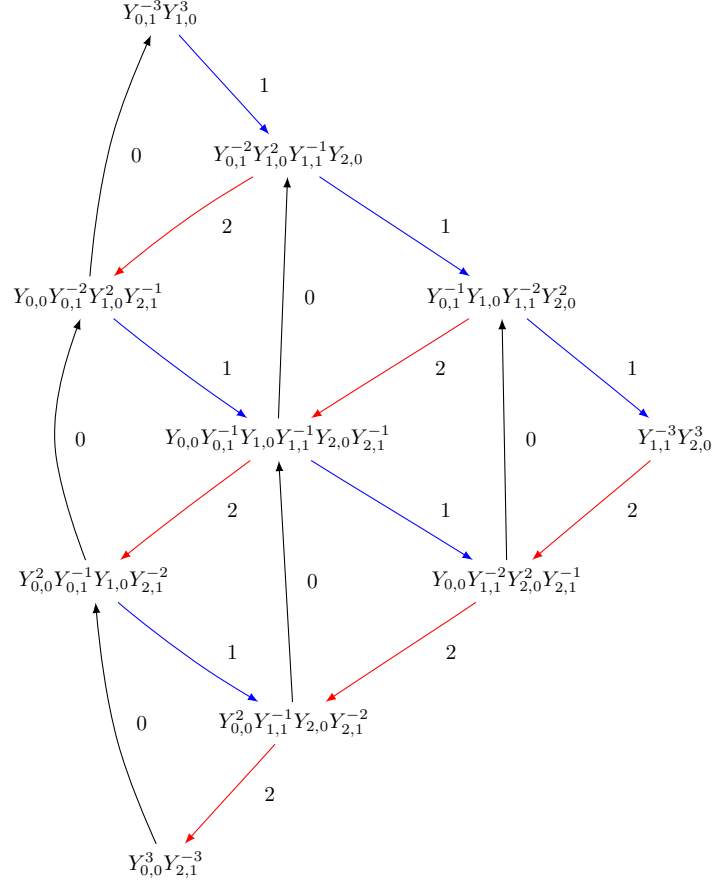
For  $i \in I$ , we have

$$\begin{aligned} f_i(\Phi(x_1, \dots, x_n)) &= f_i \left( \prod_{k=1}^n Y_{k-1,1}^{-x_k} Y_{k,0}^{x_k} \right) \\ &= f_i \left( Y_{i-1,1}^{-x_i} Y_{i,0}^{x_i} Y_{i,1}^{-x_{i+1}} Y_{i+1,0}^{x_{i+1}} \right) \prod_{k \neq i, i+1} Y_{k-1,1}^{-x_k} Y_{k,0}^{x_k} \\ &= f_i \left( Y_{i,1}^{-x_{i+1}} Y_{i,0}^{x_i} \right) Y_{i-1,1}^{-x_i} Y_{i+1,0}^{x_{i+1}} \prod_{k \neq i, i+1} Y_{k-1,1}^{-x_k} Y_{k,0}^{x_k} \\ &= \left( Y_{i,1}^{-x_{i+1}-1} Y_{i,0}^{x_i-1} Y_{i-1,1} Y_{i+1,0} \right) Y_{i-1,1}^{-x_i} Y_{i+1,0}^{x_{i+1}} \prod_{k \neq i, i+1} Y_{k-1,1}^{-x_k} Y_{k,0}^{x_k} \\ &= Y_{i-1,1}^{-x_i+1} Y_{i,0}^{x_i-1} Y_{i,1}^{-x_{i+1}-1} Y_{i+1,0}^{x_{i+1}+1} \prod_{k \neq i, i+1} Y_{k-1,1}^{-x_k} Y_{k,0}^{x_k} \\ &= \Phi(\dots, x_i - 1, x_{i+1} + 1, \dots) \\ &= \Phi(f_i(x_1, \dots, x_n)). \end{aligned}$$

where all indices are taken mod  $n$ . Similarly, we have  $e_i(\Phi(x_1, \dots, x_n)) = \Phi(e_i(x_1, \dots, x_n))$ .  $\square$

Note that we can define  $\Phi(x_1, \dots, x_n) = X_{1,0}^{x_1} X_{2,0}^{x_2} \cdots X_{n,0}^{x_n}$ , where the variables  $X_{i,k}$  are given by Equation (2.1).

**Example 3.2.** The crystal  $\mathcal{M}^{1,3}$  for  $\widehat{\mathfrak{sl}}_3$  is given by Figure 2.

FIGURE 2. The crystal  $\mathcal{M}^{1,3}$  for  $\widehat{\mathfrak{sl}}_3$ .

## 4. RELATION TO KYOTO PATH MODEL

To define  $\mathcal{M}^{1,s}$ , we considered the crystal generated from  $Y_{0,1}^{-s} Y_{1,0}^s$ . However, by shifting the monomials, we can construct an isomorphism with the tensor product. Indeed, let  $\tau_j$  be the map given by  $Y_{i,k} \mapsto Y_{i,k+j}$  for all  $i \in I$  and  $k \in \mathbb{Z}$ . Let  $\mathcal{M} \cdot \mathcal{M}' = \{m \cdot m' \mid m \in \mathcal{M}, m' \in \mathcal{M}'\}$ , where  $\cdot$  denotes the usual multiplication in  $\mathcal{M}$ , under the usual crystal operators.

**Theorem 4.1.** *Let  $j_1, j_2, \dots, j_N \in \mathbb{Z}_{\geq 0}$  be pairwise distinct. We have*

$$\prod_{k=1}^N \tau_{j_k}(\mathcal{M}^{1,s_k}) \cong \bigotimes_{k=1}^N B^{1,s_k}.$$

*Proof.* Let  $\Phi: \bigotimes_{k=1}^N B^{1,s_k} \rightarrow \prod_{k=1}^N \tau_{j_k}(\mathcal{M}^{1,s_k})$  be the map

$$\Phi(b_1 \otimes \dots \otimes b_N) = \prod_{k=1}^N \tau_{j_k}(\Phi_{s_k}(b_k)),$$

where  $\Phi_{s_k}: B^{1,s_k} \rightarrow \mathcal{M}^{1,s_k}$  is the isomorphism given by Theorem 3.1. From the combinatorial  $R$ -matrix and the commutativity of the  $Y_{i,k}$ , we can assume without loss of generality that  $j_1 < j_2 < \dots < j_N$ .

We show the claim holds by induction on  $N$ . Theorem 3.1 says this holds when  $N = 1$ . Thus assume the claim holds for  $N - 1$ .

Consider the tensor product  $B \otimes B^{1,s_N}$ , where  $B = \bigotimes_{k=1}^{N-1} B^{1,s_k}$ , and  $j_N > j_{N-1}$ . Recall from the proof of Theorem 3.1 that  $-\varepsilon_i(m)$  and  $\varphi_i(m)$  are equal to the powers of  $Y_{i,1}$  and  $Y_{i,0}$ , respectively, appearing in  $m \in \mathcal{M}^{1,s_k}$ . Let  $\psi: \Phi(B) \otimes \tau_{j_N}(\mathcal{M}^{1,s_N}) \rightarrow \Phi(B) \cdot \tau_{j_N}(\mathcal{M}^{1,s_N})$ .

Consider

$$m = \prod_{i \in I} \prod_{k \in \mathbb{Z}} Y_{i,k}^{y_{i,k}} \in \Phi(B),$$

$$m' = \prod_{i \in I} \prod_{k \in \mathbb{Z}} Y_{i,k}^{y'_{i,k}} \in \tau_{j_N}(\mathcal{M}^{1,s_N}).$$

Note that  $\psi(m \otimes m') = m \cdot m'$ . Also, for all  $i \in I$ , we have  $y_{i,p} = 0$  for all  $p > j_N$ ,  $y'_{i,p} = 0$  for all  $p < j_N$  or  $p > j_N + 1$ . Thus, we have

$$\sum_{p > k} y_{i,p} + y'_{i,p} = \begin{cases} y'_{i,j_N+1} + y'_{i,j_N} + \sum_{p > k} y_{i,p} & \text{if } k < j_N, \\ y'_{i,j_N+1} & \text{if } k = j_N, \\ 0 & \text{if } k > j_N, \end{cases} \quad (4.1a)$$

$$\sum_{p \leq k} y_{i,p} + y'_{i,p} = \begin{cases} \sum_{p \leq k} y_{i,p} & \text{if } k < j_N, \\ y'_{i,j_N} + \sum_{p \leq j_N} y_{i,p} & \text{if } k = j_N, \\ y'_{i,j_N+1} + y'_{i,j_N} + \sum_{p \leq j_N} y_{i,p} & \text{if } k > j_N. \end{cases} \quad (4.1b)$$

Note that

$$\langle h_i, \text{wt}(m') \rangle = \varphi_i(m') - \varepsilon_i(m') = y'_{i,j_N} + y'_{i,j_N+1}.$$

From Equation (4.1a), either  $y'_{i,j_N+1}$  obtains the minimum in  $\varepsilon_i(m \cdot m')$  or  $\varepsilon_i(m \cdot m') = \varepsilon_i(m) - \langle h_i, \text{wt}(m') \rangle$ . Hence, we have

$$\varepsilon_i(m \otimes m') = \max(\varepsilon_i(m'), \varepsilon_i(m) - \langle h_i, \text{wt}(m') \rangle) = \varepsilon_i(m \cdot m').$$

Next, from Equation (4.1b), either  $\varphi_i(m)$  obtains the maximum in  $\varphi_i(m \cdot m')$  or  $\varphi_i(m \cdot m') = \varphi_i(m') + \langle h_i, \text{wt}(m) \rangle$  as  $y'_{i,j_N+1} \leq 0$  and  $\sum_{s \leq j_N} y_{i,s} = \langle h_i, \text{wt}(m) \rangle$ . Hence, we have

$$\varphi_i(m \otimes m') = \max(\varphi_i(m), \varphi_i(m') + \langle h_i, \text{wt}(m) \rangle) = \varphi_i(m \cdot m').$$

It is clear that  $\text{wt}(m \otimes m') = \text{wt}(m) + \text{wt}(m') = \text{wt}(m \cdot m')$ . Thus it remains to show  $\psi$  commutes with the crystal operators.

Suppose  $k_e(m \cdot m') < j_N$ . Thus, we must have

$$-\varepsilon_i(m') = y'_{i,j_N} < y'_{i,j_N} + y'_{i,j_N+1} + \sum_{p > k} y_{i,p} = -\varepsilon_i(m) + \varphi_i(m') - \varepsilon_i(m')$$

for some  $k$  from Equation (4.1a) as  $k_e(m \cdot m')$  is the maximum index. Hence, we have  $\varphi_i(m') > \varepsilon_i(m)$ . Similarly, if  $k_e(m \cdot m') = j_N$ , then we have

$$y'_{i,j_N+1} = -\varepsilon_i(m \cdot m') = -\varepsilon_i(m') \geq -\varepsilon_i(m) + \varphi_i(m') - \varepsilon_i(m')$$

from Equation (4.1a). Thus we have  $\varphi_i(m') \leq \varepsilon_i(m)$ . Therefore, we have

$$e_i(\psi(m \otimes m')) = \psi(e_i(m \otimes m')) = \psi(e_i m \otimes m').$$

Suppose  $k_f(m \cdot m') < j_N$ . Thus we must have

$$\varphi_i(m') + \varphi_i(m) - \varepsilon_i(m) = y'_{i,j_N} + \langle h_i, \text{wt}(m) \rangle \leq \varphi_i(m \cdot m')$$

from Equation (4.1b) and that  $k_f(m \cdot m')$  is the minimal index. Thus, we have  $\varphi_i(m') \leq \varepsilon_i(m)$ . Similarly, if  $k_f(m \cdot m') = j_N$ , then we have

$$\varphi_i(m \cdot m') = \varphi_i(m') + \varphi_i(m) - \varepsilon_i(m) > \varphi_i(m)$$

as this is the minimal index such that  $\varphi_i(m \cdot m')$  is achieved from Equation (4.1b). Hence, we have  $\varphi_i(m') > \varepsilon_i(m)$ . Therefore, we have

$$f_i(\psi(m \otimes m')) = \psi(f_i(m \otimes m')) = \psi(f_i m \otimes m').$$

□

**Theorem 4.2.** *Let  $\lambda \in P^+$  be a level  $s$  weight. Let  $\Xi: B(\lambda) \rightarrow \mathcal{M}(\lambda)$  be the canonical isomorphism. Let  $\Psi: B(\lambda) \rightarrow B^{1,s} \otimes B(\mu)$  be the isomorphism from Equation (2.4). Let  $\Phi: B^{1,s} \otimes B(\mu) \rightarrow \mathcal{M}(\lambda)$  denote the map given by*

$$\Phi(b \otimes b') = \Phi_s(b) \cdot \tau_1(\Phi_\mu(b')),$$

where  $\Phi_s: B^{1,s} \rightarrow \mathcal{M}^{1,s}$  and  $\Phi_\mu: B(\mu) \rightarrow \mathcal{M}(\mu)$  be the isomorphisms from Theorem 3.1 and Theorem 2.3, respectively. Then the diagram

$$\begin{array}{ccc} B(\lambda) & \xrightarrow{\Psi} & B^{1,s} \otimes B(\mu) \\ & \searrow \Xi & \downarrow \Phi \\ & & \mathcal{M}(\lambda) \end{array}$$

commutes.

*Proof.* Note that the crystal operators imply that any  $m \in \mathcal{M}(\mu)$  does not contain any  $Y_{i,k}$  with  $k \in \mathbb{Z}_{<0}$ . To show the diagram commutes, it is sufficient to show that for  $b = b^\lambda \otimes u_\mu$ , we have

$$\Phi(b) = \Phi_s(b^\lambda) \cdot \tau_1(\Phi_\mu(u_\mu)) = Y_\lambda$$

as the proof of Theorem 4.1 implies the crystal operators commute with  $\Phi$ .

From the definition of  $\Psi$ , we have  $\varphi(b^\lambda) = \lambda$  and  $\varepsilon(b^\lambda) = \mu$ . Next, note that we have

$$\begin{aligned} \tau_1(\Phi_\mu(u_\mu)) &= \tau_1(Y_\mu) = \prod_{i \in I} Y_{i,1}^{(h_i, \mu)} = \prod_{i \in I} Y_{i,1}^{\varepsilon_i(b^\lambda)}, \\ \Phi_s(b^\lambda) &= \prod_{i \in I} Y_{i,1}^{-\varepsilon_i(b^\lambda)} Y_{i,0}^{\varphi_i(b^\lambda)}. \end{aligned}$$

Hence, we have

$$\Phi(b) = \prod_{i \in I} Y_{i,1}^{\varepsilon_i(b^\lambda)} \prod_{i \in I} Y_{i,1}^{-\varepsilon_i(b^\lambda)} Y_{i,0}^{\varphi_i(b^\lambda)} = Y_\lambda.$$

□

**Remark 4.3.** Note that Theorem 4.2 implies that there exists an isomorphism  $\mathcal{M}(\lambda) \cong \mathcal{M}^{1,s} \otimes \mathcal{M}(\mu)$ . This is also implied by the proof of Theorem 4.2, which gives an alternative proof of the existence of the isomorphism given by Equation (2.4). Moreover, this also recovers Theorem 2.5.

**Example 4.4.** Consider type  $U'_q(\widehat{\mathfrak{sl}}_5)$ . The ground-state path is given by

$$\boxed{5} \otimes \boxed{4} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{5} \otimes \boxed{4} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{1} \otimes \cdots.$$

Therefore, by iterating the isomorphism  $\Phi$  from Theorem 4.2, we have

$$\begin{aligned} \Phi(u_{\Lambda_0}) &= Y_{0,0} Y_{4,1}^{-1} \tau_1(Y_{4,0} Y_{3,1}^{-1}) \tau_2(Y_{3,0} Y_{2,1}^{-1}) \tau_3(Y_{2,0} Y_{1,1}^{-1}) \tau_4(Y_{1,0} Y_{0,1}^{-1}) \tau_5(Y_{0,0} Y_{4,1}^{-1}) \cdots \\ &= Y_{0,0} Y_{4,1}^{-1} Y_{4,1} Y_{3,2}^{-1} Y_{3,2} Y_{2,3}^{-1} Y_{2,3} Y_{1,4}^{-1} Y_{1,4} Y_{0,5}^{-1} Y_{0,5} Y_{4,6}^{-1} \cdots \\ &= Y_{0,0} \end{aligned}$$

By restricting the tensor product given above to a finite number of factors and not including the highest weight crystal, we have

$$\mathcal{M}(Y_{-m,m}^{-s} Y_{0,0}^s) \cong (B^{1,s})^{\otimes m}.$$

5. COHERENT LIMIT AND  $B(\infty)$ 

We describe the coherent limit  $B_\infty$  in terms of Nakajima monomials.

Let  $\mathcal{M}_\infty$  denote the coherent limit of  $\mathcal{M}^{1,s}$ . We explicitly give the crystal structure on  $\mathcal{M}_\infty$  as follows. We define

$$k'_e(m) = \begin{cases} 0 & \text{if } k_e(m) \text{ is undefined,} \\ k_e(m) & \text{otherwise,} \end{cases}$$

$$k'_f(m) = \begin{cases} 0 & \text{if } k_f(m) \text{ is undefined,} \\ k_f(m) & \text{otherwise,} \end{cases}$$

and consider  $\mathcal{M}_\infty$  as the closure of  $\mathbf{1}$  under the modified crystal operators

$$e'_i(m) = mA_{i,k'_e(m)-1}, \quad f'_i(m) = mA_{i,k'_f(m)-1}^{-1}.$$

**Theorem 5.1.** *The map  $\Psi: B_\infty \rightarrow \mathcal{M}_\infty$  given by*

$$(x_1, \dots, x_n) \mapsto X_{1,0}^{x_1} \cdots X_{n,0}^{x_n}$$

*is a  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystal isomorphism.*

*Proof.* This is similar to the proof of Theorem 3.1.  $\square$

**Theorem 5.2.** *The map*

$$\Theta: \mathcal{M}_\infty \otimes \mathcal{M}(\infty) \rightarrow \mathcal{M}(\infty)$$

*given by  $m \otimes m' \mapsto m \cdot \tau_1(m')$  is a  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystal isomorphism.*

*Proof.* Let  $\mathbf{1}^\infty$  denote the highest weight element of  $\mathcal{M}(\infty)$  (which is the constant monomial  $Y_0$ ). Note that  $\tau_1(\mathbf{1}^\infty) = \mathbf{1}^\infty$ . The proof is similar to the proof of Theorem 4.1 and Theorem 4.2.  $\square$

Note that Theorem 5.2 is the Nakajima monomial version of Theorem 2.6. Hence, we recover the path model for  $B(\infty)$  and [KKS07, Thm. 5.1].

## 6. EXTENSIONS AND CONNECTIONS

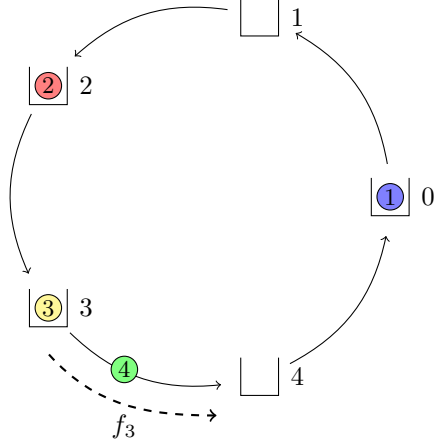
In this section, we give some connections of our results to other crystal models. We also give (potential) extensions of our results to  $B^{r,1}$  and other types.

**6.1. Single columns.** We can only do  $B^{1,s}$  in type  $A_n^{(1)}$  using Kashiwara's crystal structure without a quotient precisely because the Dynkin diagram contains a directed cycle. We never repeat an  $f_i$  along any path from  $b \in B^{1,s}$  back to itself which matches that of the path in the Dynkin diagram. So we never increase our indices. This is false in all other orientations of the Dynkin diagram. However, we are able to construct  $B^{r,1}$  using a modification of Nakajima monomials.

We note that elements of  $B^{r,1}$  can be described by  $(x_1, \dots, x_n)$  such that  $0 \leq x_i \leq 1$  and  $\sum_{i=1}^n x_i = r$  with the crystal structure the same as the vector representation given in Section 2.3. Hence, we have the following fact, which does not appear in the literature as far as we are aware, but is likely known to experts.

**Proposition 6.1.** *There exists a crystal embedding  $B^{r,1} \rightarrow B^{1,r}$ .*

Therefore, we can quotient our monomials by  $X_{i,k}^2$  and obtain a description for  $B^{r,1}$  in terms of Nakajima monomials. Explicitly, given a monomial  $m$ , define modified crystal operator  $\overline{f}_i$  by  $\overline{f}_i(m) = f_i(m)$  if  $f_i(m)$  does not contain an  $X_{i,k}^2$  for some  $(i, k) \in I \times \mathbb{Z}$  and  $\overline{f}_i(m) = 0$  otherwise. The definition of  $\overline{e}_i$  is defined similarly by replacing  $f_i$  with  $e_i$ . Let  $\overline{\mathcal{M}}(m)$  denote the closure of  $m$  under  $\overline{e}_i$  and  $\overline{f}_i$ .

FIGURE 3. The model of  $B^{1,4}$  for  $U_q(\widehat{\mathfrak{sl}}_5)$  using balls in bins.

**Proposition 6.2.** *For any  $k \in \mathbb{Z}$ , we have*

$$B^{r,1} \cong \overline{\mathcal{M}} \left( \prod_{i=1}^r X_{i,k} \right)$$

as  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystals.

Recall that the highest weight  $U_q(\mathfrak{sl}_n)$ -crystal  $B(\overline{\Lambda}_r)$  corresponding to  $B^{r,1}$  can be constructed from the exterior product of the basic representation  $B(\overline{\Lambda}_1)$  of  $U'_q(\mathfrak{sl}_n)$ . Therefore, it is more natural to consider the variables  $X_{i,k}$  as anticommuting variables to describe  $B(\overline{\Lambda}_r)$  using Nakajima monomials (up to a sign).

**6.2. Relations to other models.** Recall the abacus model from [Tin08], where we model  $B(\lambda)$  by  $\ell$  strings, with  $\ell$  being the level of  $\lambda$ , with beads and the crystal operators act by moving a bead. Fix some abacus configuration  $\psi$ . Let  $\psi_i^k$  denote the number of beads such that it is the  $k$ -th bead in a string in position  $i$  modulo  $n$ . Define

$$\Phi(\psi) = \prod_{i \in I} \prod_{k=0}^{\infty} X_{i,k}^{\psi_i^k},$$

and we note that this is just a translation of the map  $J$  described in [Tin08, Thm. 5.1] in terms of our Nakajima monomial model. Therefore,  $\Phi$  is a  $U'_q(\widehat{\mathfrak{sl}}_n)$ -crystal isomorphism.

Additionally, a bijection between Nakajima monomials and quiver varieties was constructed in [ST14, Thm.8.5]. Indeed, we realize  $\widehat{B}^{1,1}$  by the quiver variety represented by an infinite string of arrows

$$[b] = \cdots \longrightarrow (b-2) \longrightarrow (b-1) \longrightarrow b.$$

where  $f_i[b]$  adds an arrow  $b \rightarrow b+1$  if  $b \equiv i \pmod n$  and is 0 otherwise. We can consider this as an infinite number of bins with a single ball, where the crystal operator  $f_i$  moves the ball if it is in the  $b$ -th bin, where  $b \equiv i \pmod n$ . Thus, we can model  $B^{1,1}$  by considering the bins in a cycle of length  $n$ . Furthermore, from the projection of  $\widehat{B}^{1,1}$  onto  $B^{1,1}$ , we expect a description of  $B^{1,1}$  using cyclic quiver varieties.

We can extend this ball-bin model into  $B^{1,s}$  by using  $s$ -colored balls, where  $f_i$  moves the largest colored ball in bin  $i$ . See Figure 3 for an example. Similarly, we can construct  $\widehat{B}^{1,s}$  by starting with an  $s$ -fold stack of  $[0]$  and  $f_i$  adding an arrow to the bottom string (if possible).

**Example 6.3.** Consider  $\widehat{B}^{1,3}$  for  $U'_q(\widehat{\mathfrak{sl}}_4)$ . We can represent the element  $(1, 0, 2, 0)(3)$  by the infinite strings

$$[13, 15, 15] = \begin{array}{cccccccc} \cdots & \longrightarrow & 11 & \longrightarrow & 12 & \longrightarrow & 13 & \\ \cdots & \longrightarrow & 11 & \longrightarrow & 12 & \longrightarrow & 13 & \longrightarrow & 14 & \longrightarrow & 15 & . \\ \cdots & \longrightarrow & 11 & \longrightarrow & 12 & \longrightarrow & 13 & \longrightarrow & 14 & \longrightarrow & 15 \end{array}$$

Then, we have

$$\begin{array}{l} f_1[13, 15, 15] = [14, 15, 15] = \begin{array}{cccccccc} \cdots & \longrightarrow & 11 & \longrightarrow & 12 & \longrightarrow & 13 & \longrightarrow & 14 \\ \cdots & \longrightarrow & 11 & \longrightarrow & 12 & \longrightarrow & 13 & \longrightarrow & 14 & \longrightarrow & 15 & , \\ \cdots & \longrightarrow & 11 & \longrightarrow & 12 & \longrightarrow & 13 & \longrightarrow & 14 & \longrightarrow & 15 \end{array} \\ f_3[13, 15, 15] = [13, 15, 16] = \begin{array}{cccccccc} \cdots & \longrightarrow & 11 & \longrightarrow & 12 & \longrightarrow & 13 & \longrightarrow & 14 \\ \cdots & \longrightarrow & 11 & \longrightarrow & 12 & \longrightarrow & 13 & \longrightarrow & 14 & \longrightarrow & 15 \\ \cdots & \longrightarrow & 11 & \longrightarrow & 12 & \longrightarrow & 13 & \longrightarrow & 14 & \longrightarrow & 15 & \longrightarrow & 16 \end{array} . \end{array}$$

We can also realize  $B^{r,1}$  by considering the  $n$  bins placed on a cycle, but now each bin can hold at most one ball. The crystal operators  $f_i$  acts by moving a ball from bin  $i$  to bin  $i + 1$  if possible, and otherwise is defined as 0.

We recall that bijections between the partition model, the abacus model, and cylindric plane partition model for  $B(\lambda)$  are given in [Tin08]. A bijection  $\Xi$  between the Nakajima monomial model and the partition model for  $B(\Lambda_i)$  was given in [Tin10]. Thus, our bijection  $\Phi$  connects these two results, and moreover, generalizes the bijection  $\Xi$  to higher levels.

**6.3.  $R$ -matrix kernel and other types.** From Theorem 4.1, we can construct the tensor product  $B^{1,1} \otimes B^{1,1}$  by considering the product  $\mathcal{M}^{1,1}$  with its shifted version  $\tau_1(\mathcal{M}^{1,1})$ . However, if we want to avoid the shift, we can still construct  $\mathcal{M}^{1,1} \otimes \mathcal{M}^{1,1}$  by multiplication but having multiplication twisted by a generic parameter  $t$ . This is a special case of the results of [Nak03b, Nak03a, Nak04, Her04, KN12] expressed in terms of Kashiwara's variation of Nakajima monomials. The kernel of the appropriate parameterized  $R$ -matrix is generated by

$$K = \{m \otimes m' - tm' \otimes m \mid m \neq m' \in \mathcal{M}^{1,1}\}.$$

Note that we can construct the elements of  $K$  by taking all twisted commutators for the  $q, t$ -character, where we consider  $\otimes$  as multiplication.

We also recall there is a statistic called *energy* on tensor products of KR crystals [KKM<sup>+</sup>92a, KKM<sup>+</sup>92b]. We denote the energy of  $b$  by  $E(b)$ . If we take the quotient of  $\mathcal{M}^{1,1} \otimes \mathcal{M}^{1,1}$  where we consider elements as  $t^{E(b)}b$ , the quotient by  $K$  results in  $\mathcal{M}^{1,2}$ . This can also be extended to  $\mathcal{M}^{1,s}$ .

**Example 6.4.** Consider the  $U'_q(\widehat{\mathfrak{sl}}_3)$ -crystals  $\mathcal{M}^{1,1} \otimes \mathcal{M}^{1,1}$  and  $\mathcal{M}^{1,2}$  (see Figure 4). The graded  $q$ -character of  $\mathcal{T} = \mathcal{M}^{1,1} \otimes \mathcal{M}^{1,1}$  is

$$\begin{aligned} \sum_{m \otimes m' \in \mathcal{T}} t^{E(m \otimes m')} m \cdot m' &= Y_{0,1}^{-2} Y_{1,0}^2 + (t+1) Y_{0,0} Y_{1,1}^{-1} Y_{2,0} Y_{2,1}^{-1} + (t+1) Y_{0,0} Y_{0,1}^{-1} Y_{1,0} Y_{2,1}^{-1} \\ &\quad + (t+1) Y_{0,1}^{-1} Y_{1,0} Y_{1,1}^{-1} Y_{2,0} + Y_{1,1}^{-2} Y_{2,0}^2 + Y_{0,0}^2 Y_{2,1}^{-2}. \end{aligned}$$

By considering the (graded) decomposition into  $U_q(\mathfrak{sl}_3)$ -crystals, we get the same decomposition (after  $t \mapsto t^2$ ) as computed by [Nak03b, Nak03a, Nak04, Her04] (note that we are using different Nakajima monomials). This correspondence is an example of the results of [KN12]. However, if we take the quotient of the graded  $q$ -character generated by  $K$  (with replacing  $\otimes$  by  $\cdot$ ), we obtain

$$Y_{1,1}^{-2} Y_{2,0}^2 + Y_{0,0} Y_{1,1}^{-1} Y_{2,0} Y_{2,1}^{-1} + Y_{0,0} Y_{0,1}^{-1} Y_{1,0} Y_{2,1}^{-1} + Y_{0,1}^{-2} Y_{1,0}^2 + Y_{0,0}^2 Y_{2,1}^{-2} + Y_{0,1}^{-1} Y_{1,0} Y_{1,1}^{-1} Y_{2,0},$$

which is the graded  $q$ -character of  $\mathcal{M}^{1,2}$ .



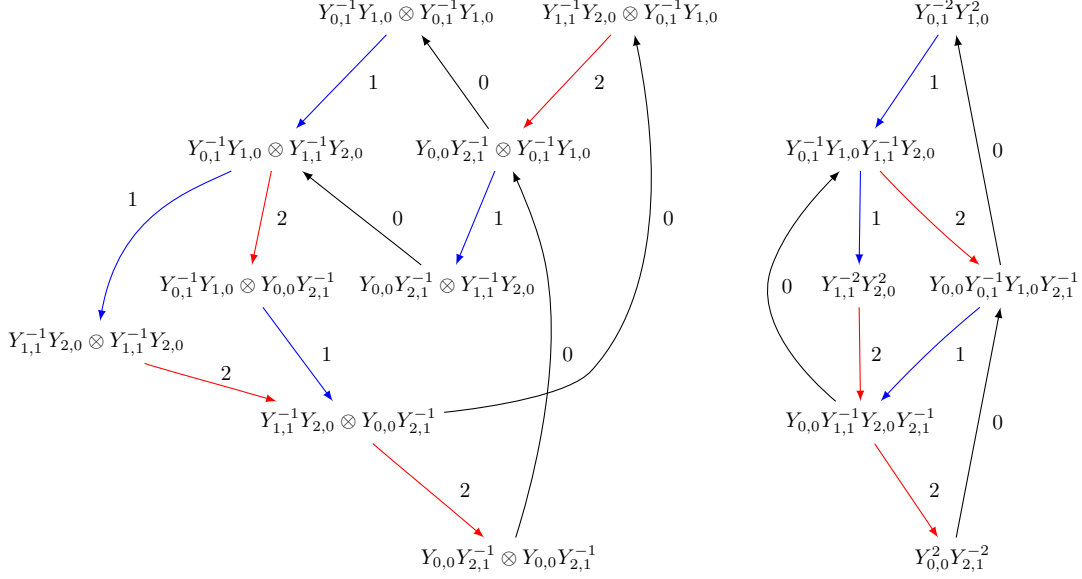


FIGURE 4. The  $U'_q(\widehat{\mathfrak{sl}}_3)$ -crystal  $\mathcal{M}^{1,1} \otimes \mathcal{M}^{1,1}$  (left) and  $\mathcal{M}^{1,2}$  (right).

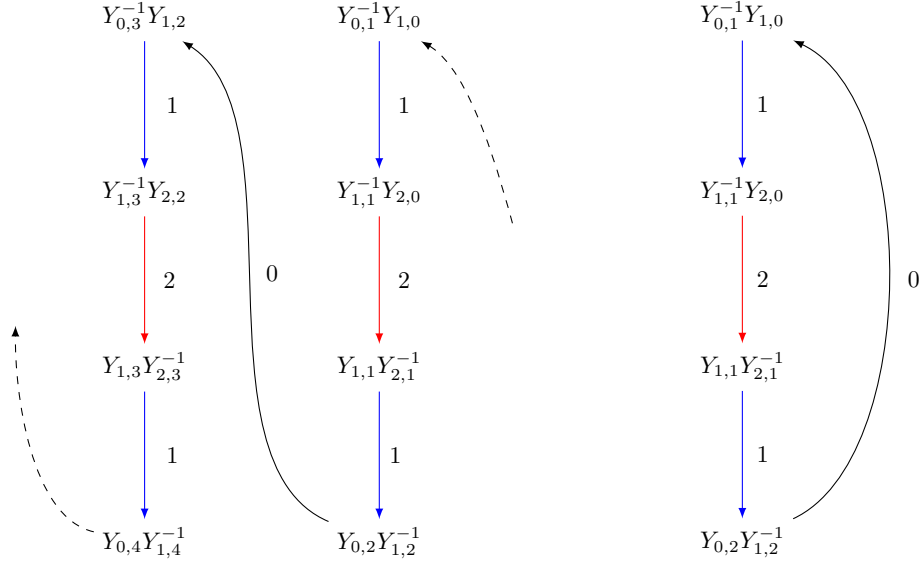


FIGURE 5. A portion of the level-zero crystal  $\mathcal{M}(Y_{0,1}^{-1}Y_{1,0})$  (left) and  $\mathcal{M}^{1,1}$  constructed from  $\mathcal{M}(Y_{0,1}^{-1}Y_{1,0})/\kappa$  (right) in type  $C_2^{(1)}$ .

Our construction works because  $B^{1,1}$  in type  $A_n^{(1)}$  “follows” our orientation of the Dynkin diagram. More generally, in order to construct  $\mathcal{M}^{1,1}$ , we need to take a quotient of a level-zero crystal  $\mathcal{M}(m)$ , where  $m$  is some monomial whose weight is of level 0. For example, in type  $C_2^{(1)}$  with  $c_{ij} = 1$  if  $i < j$  and 0 otherwise, we can construct  $\mathcal{M}^{1,1}$  as a quotient of  $\mathcal{M}(Y_{0,1}^{-1}Y_{1,0})$  by an automorphism  $\kappa$  given by  $Y_{i,k}^{-1}Y_{i',k'}^{-1} \mapsto Y_{i,k-2}^{-1}Y_{i',k'-2}^{-1}$  as in Figure 5. However,  $\mathcal{M}^{1,1} \cdot \mathcal{M}^{1,1}$  is not isomorphic to  $B^{1,2}$  as the

former has 10 elements and the latter has 11. This also agrees with instead taking  $\mathcal{M}(Y_{0,1}^{-2}Y_{1,0}^2)$  and then taking the quotient by  $\kappa$ .

Additionally, the quotient of the kernel of the parameterized  $R$ -matrix corresponding to  $B^{1,1} \otimes B^{1,1}$  gives rise to a twisted commutator in type  $A_n^{(1)}$ . Therefore, we expect that by considering the variables in  $\mathcal{M}^{1,1}$  as non-commuting variables and then taking an appropriate quotient by the kernel of the  $R$ -matrix, we could construct general  $B^{r,s}$  for all affine types. Furthermore, by relating the extra parameter in the kernel to energy, we expect to recover the results of [KN12] that relate the  $q, t$ -characters of standard modules to those of simple modules. Indeed, for the type  $C_2^{(1)}$  case considered above, if we add the grading by energy in  $\mathcal{M}^{1,1} \cdot \mathcal{M}^{1,1}$ , then we obtain 14 elements. However, if we additionally quotient by the kernel of the  $R$ -matrix, we can construct a Nakajima monomial realization of  $B^{1,2}$ .

## APPENDIX A. EXAMPLES WITH SAGEMATH

We give some examples using SAGEMATH [Dev16] using the crystal of Nakajima monomials implemented by Ben Salisbury and Arthur Lubovsky.

We construct  $B^{1,2}$  for  $\widehat{\mathfrak{sl}}_5$  as given in Section 4:

```
sage: P = RootSystem(['A', 4, 1]).weight_lattice(extended=True)
sage: La = P.fundamental_weights()
sage: c = matrix([[0, 1, 1, 1, 0], [0, 0, 1, 1, 1], [0, 0, 0, 1, 1],
.....:           [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]])
sage: c
[0 1 1 1 0]
[0 0 1 1 1]
[0 0 0 1 1]
[0 0 0 0 1]
[1 0 0 0 0]
sage: M = crystals.NakajimaMonomials(La[1]-La[0], c=c)
sage: x = M({(0,1):-2, (1,0):2}, {})
sage: S = x.subcrystal()
sage: view(S, tightpage=True)
sage: K = crystals.KirillovReshetikhin(['A', 4, 1], 1, 2)
sage: K.digraph().is_isomorphic(S.digraph(), edge_labels=True)
True
sage: x = M({(0,1):-2, (1,0):2, (0,2):-1, (1,1):1}, {})
sage: K1 = crystals.KirillovReshetikhin(['A', 4, 1], 1, 1)
sage: T = tensor([K, K1])
sage: S = x.subcrystal()
sage: T.digraph().is_isomorphic(S.digraph(), edge_labels=True)
True
```

Next we construct  $B^{1,1} \otimes B^{1,2}$  for  $\widehat{\mathfrak{sl}}_3$ :

```
sage: P = RootSystem(['A', 2, 1]).weight_lattice(extended=True)
sage: La = P.fundamental_weights()
sage: c = matrix([[0, 1, 0], [0, 0, 1], [1, 0, 0]])
sage: M = crystals.NakajimaMonomials(La[1]-La[0], c=c)
sage: x = M({(0,1):-1, (0,2):-2, (1,0):1, (1,1):2}, {})
sage: S = x.subcrystal()
sage: view(S, tightpage=True)
sage: K1 = crystals.KirillovReshetikhin(['A', 2, 1], 1, 1)
sage: K2 = crystals.KirillovReshetikhin(['A', 2, 1], 1, 2)
sage: T = tensor([K1, K2])
sage: T.digraph().is_isomorphic(S.digraph(), edge_labels=True)
True
```

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